3.6 Factorising Arithmetic Functions.

Question If Dirichlet convolution "combines" arithmetic functions can we factor a given function into a convolution of "simpler" functions?

The same method as was used to show that $\zeta(s)$ has a Dirichlet Product can be used to prove the following.

Theorem 3.28 *If f is multiplicative and $D_f(s)$ is absolutely convergent at $s_0 \in \mathbb{C}$ then, for all $s : \operatorname{Re} s > \operatorname{Re} s_0$, the Euler Product*

$$\prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \frac{f(p^3)}{p^{3s}} + ...\right)$$

converges to $D_f(s)$.

Proof Left to student, but see appendix if stuck. ■

If f is multiplicative then Theorem 3.28 gives

$$D_f(s) = \prod_p \left(\sum_{\ell \geq 0} \frac{f(p^\ell)}{p^{\ell s}}\right), \tag{10}$$

for $\operatorname{Re} s > \operatorname{Re} s_0$, since $f(p^0) = f(1) = 1$. If, further, f is **completely** multiplicative then

$$D_f(s) = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}$$

for $\operatorname{Re} s > \operatorname{Re} s_0$ and as long as $|f(p)/p^s| < 1$ for all primes p.

The idea of this method of factorisation is to write the Dirichlet Series as an Euler product and factor each term in the product.

In all our examples $f(p^\ell)$ will **not** depend on p, only ℓ, so we can write $a_\ell = f(p^\ell)$ for $\ell \geq 0$. Write $y = 1/p^s$ and the series within (10) becomes

$$\sum_{\ell \geq 0} a_\ell y^\ell. \tag{11}$$

The aim of this method is to write this series as product and quotient of terms of the form $1 - y^m$ for various integers $m \geq 1$. For if we have a factor of the form $(1 - y^m)^{-1}$, replacing y by $1/p^s$ we find a factor of the right hand side of (10) of

$$\prod_p \left(1 - \frac{1}{p^{ms}}\right)^{-1} = \zeta(ms).$$

Further, if the sum of (11) contains a factor of $1 - y^k$ for some $k \geq 1$, then on replacing y by $1/p^s$ we find a factor of the right hand side of (10) of

$$\prod_p \left(1 - \frac{1}{p^{ks}}\right) = \frac{1}{\zeta(ks)}.$$

As a way of illustrating this method:

Recall that Q_k is the characteristic function of the k-free integers.

Example 3.29 *Show that*

$$\sum_{n=1}^{\infty} \frac{Q_k(n)}{n^s} = \frac{\zeta(s)}{\zeta(ks)}$$

for $\operatorname{Re} s > 1$.

Solution The function Q_k is multiplicative so, without yet considering the regions of convergence for the Dirichlet Series, we have the Euler Product

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{Q_k(n)}{n^s} &= \prod_p \left(1 + \frac{Q_k(p)}{p^s} + \frac{Q_k(p^2)}{p^{2s}} + \frac{Q_k(p^3)}{p^{3s}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{(k-1)s}}\right),\end{aligned}$$

since $Q_k(p^\ell) = 0$ for all $\ell \geq k$, and $= 1$ elsewhere. Write $y = 1/p^s$ when each bracket is a finite geometric sum of the form

$$1 + y + y^2 + \cdots + y^{k-1} = \frac{1 - y^k}{1 - y}.$$

Therefore

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{Q_k(n)}{n^s} &= \prod_p \left(\frac{1 - 1/p^{ks}}{1 - 1/p^s}\right) \\ &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \left(\prod_p \left(1 - \frac{1}{p^{ks}}\right)^{-1}\right)^{-1} \\ &= \frac{\zeta(s)}{\zeta(ks)}, \qquad (12)\end{aligned}$$

having used (7).

We can now consider convergence. Since the ζ-functions on the right hand side are absolutely convergent in $\operatorname{Re} s > 1$, the final result is valid in this half plane. ■

Note that

$$\frac{1}{\zeta(ks)} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^{ks}} = \sum_{m=1}^{\infty} \frac{\mu(m)}{(m^k)^s} = \sum_{\substack{n=1 \\ n=m^k}}^{\infty} \frac{\mu(m)}{n^s} = \sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^s}$$

where μ_k is given by

Definition 3.30

$$\mu_k(n) = \begin{cases} \mu(m) & \text{if } n = m^k, \\ 0 & \text{otherwise.} \end{cases}$$

The Möbius Function is μ_1.

Example 3.29 shows that

$$D_{Q_k}(s) = \zeta(s) \frac{1}{\zeta(ks)} = D_1(s)\, D_{\mu_k}(s) = D_{1*\mu_k}(s)\,,$$

for $\operatorname{Re} s > 1$. This 'suggests'

Example 3.31 *For all $k \geq 2$, $Q_k = 1 * \mu_k$.*

Solution Since Q_k, 1 and μ_k are all multiplicative it suffices to prove equality on prime powers. Consider

$$1 * \mu_k(p^a) = \sum_{0 \leq r \leq a} \mu_k(p^r)\,. \tag{13}$$

The terms $\mu_k(p^r)$ can only be non-zero if $k|r$. And if $k|r$, so $r = k\ell$ for some ℓ, we have $\mu_k(p^r) = \mu\left(p^\ell\right)$ which is only non-zero when $\ell = 0$ or 1. Thus $\mu_k(p^r)$ is only non-zero when $r = 0$ or k. Therefore, if $a < k$ then the sum in (13) contains only one non-zero term, $\mu_k(p^0) = 1$. If $a \geq k$ then the sum contains two non-zero terms

$$\mu_k\left(p^0\right) + \mu_k\left(p^k\right) = 1 + \mu(p) = 1 - 1 = 0.$$

Hence

$$1 * \mu_k(p^a) = \left\{ \begin{array}{ll} 1 & \text{if } a < k \\ 0 & \text{if } a \geq k \end{array} \right\} = Q_k(p^a)\,.$$

■

Note that when $k = 1$ we have seen that $Q_1 = \delta$, while $\mu_1 = \mu$ and so $Q_k = 1 * \mu_k$ reduces down to the Möbius inversion $\delta = 1 * \mu$.

The most important case of this example is $k = 2 : Q_2 = 1 * \mu_2$, which will be seen many times.

Example 3.32 *Show, by looking at Euler Product of the Dirichlet Series on the left, that*

$$\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}$$

for $\operatorname{Re} s > 1$.

Solution The left hand side has the Euler product

$$\prod_p \left(1 + \frac{2}{p^s} + \frac{2}{p^{2s}} + \frac{2}{p^{3s}} + \frac{2}{p^{4s}} +\right).$$

For $|y| < 1$,

$$\begin{aligned} 1 + 2y + 2y^2 + 2y^3 + ... &= 1 + 2y\left(1 + y + y^2 + ...\right) \\ &= 1 + \frac{2y}{1-y} \qquad \text{on summing the geometric series} \\ &= \frac{1+y}{1-y} \\ &= \frac{1+y}{1-y} \times \frac{1+y}{1-y} = \frac{1-y^2}{(1-y)^2}. \end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \prod_p \frac{1 - 1/p^{2s}}{(1 - 1/p^s)^2} = \frac{\zeta^2(s)}{\zeta(2s)}.$$

■

The result on 2^ω gives

$$D_{2^\omega}(s) = \zeta(s)\,\zeta(s)\,\frac{1}{\zeta(2s)} = D_1(s)\,D_1(s)\,D_{\mu_2}(s) = D_{1*1*\mu_2}(s)\,,$$

for $\operatorname{Re} s > 1$. This 'suggests'

Example 3.33

$$2^{\omega} = 1 * 1 * \mu_2.$$

Solution See Problem Sheet.

This can be combined with the definition $d = 1*1$ or the result $Q_2 = 1*\mu_2$ to give

$$2^{\omega} = d * \mu_2 = 1 * Q_2. \tag{14}$$

There are many such connections between Arithmetic functions, some of which are the content of questions on the Problem Sheet and all are collected on a page on the Course web site.

3.7 The decomposition of $\mathbf{d^2}$

For an example in the next Section we need the decomposition of d^2.

Example 3.34

$$D_{d^2}(s) = \frac{\zeta^4(s)}{\zeta(2s)},$$

for $\operatorname{Re} s > 1$.

Solution We note that d^2 is a multiplicative function and $d^2(p^a) = (a+1)^2$ on prime powers. So the Dirichlet Series of d^2 has the Euler Product

$$D_{d^2}(s) = \sum_{n=1}^{\infty} \frac{d^2(n)}{n^s} = \prod_p \left(1 + \frac{4}{p^s} + \frac{9}{p^{2s}} + \frac{16}{p^{3s}} + \frac{25}{p^{4s}} + ...\right),$$

for $\operatorname{Re} s > 1$. To sum the series

$$S = 1 + 4y + 9y^2 + 16y^3 + 25y^4 + ... + (a+1)^2 y^a + ...,$$

for $|y| < 1$ consider, with not justification,

$$\begin{aligned}
S &= \frac{d}{dy}\left(y + 2y^2 + 3y^3 + 4y^5 + ...\right) \\
&= \frac{d}{dy}\left(y\left(1 + 2y + 3y^2 + 4y^3 + ...\right)\right) \\
&= \frac{d}{dy}\left(y\frac{d}{dy}\left(y + y^2 + y^3 + y^4 + ...\right)\right) \\
&= \frac{d}{dy}\left(y\frac{d}{dy}\frac{y}{1-y}\right), \quad \text{on summing the geometric series,} \\
&= \frac{d}{dy}\left(\frac{y}{(1-y)^2}\right) \\
&= \frac{1+y}{(1-y)^3}.
\end{aligned}$$

Since we haven't justified the integrating and differentiating of infinite series term-by-term you need to check this result by expanding $(1+y)(1-y)^{-3}$ and getting the series you started with.

We are not quite finished for the formula for the sum needs to be written as a product and quotient of terms of the form $1 - y^m$, i.e. with a negative sign. So

$$S = \frac{1+y}{(1-y)^3} = \frac{1+y}{(1-y)^3} \times \frac{1-y}{1-y} = \frac{1-y^2}{(1-y)^4}.$$

Using this in each factor of the Euler Product for $D_{d^2}(s)$ gives

$$\begin{aligned}
D_{d^2}(s) &= \prod_p \frac{1 - 1/p^{2s}}{(1 - 1/p^s)^4} = \left(\prod_p \left(1 - \frac{1}{p^s}\right)^{-1}\right)^4 \left(\prod_p \left(1 - \frac{1}{p^{2s}}\right)^{-1}\right)^{-1} \\
&= \frac{\zeta^4(s)}{\zeta(2s)},
\end{aligned}$$

for $\operatorname{Re} s > 1$. ■

From this,

$$D_{d^2}(s) = \frac{\zeta^4(s)}{\zeta(2s)} = \zeta^4(s)\,\frac{1}{\zeta(2s)} = D_1^4(s)\,D_{\mu_2}(s) = D_{1*1*1*1*\mu_2}(s).$$

for $\operatorname{Re} s > 1$. This 'suggests' the decomposition $d^2 = 1 * 1 * 1 * 1 * \mu_2$ or, because of Example 3.33, $d^2 = 1 * 1 * 2^\omega$. We prove this in two stages.

Example 3.35 *For all* $n \geq 1$, $1 * 2^\omega(n) = d(n^2)$.

Solution Since both sides are multiplicative it suffices to check equality on prime powers.

$$
\begin{aligned}
(1 * 2^\omega)(p^r) &= \sum_{a+b=r} 2^{\omega(p^b)} = \sum_{0 \leq b \leq r} 2^{\omega(p^b)} = 2^{\omega(p^0)} + \sum_{1 \leq b \leq r} 2^{\omega(p^b)} \\
&= 2^0 + \sum_{1 \leq b \leq r} 2 = 1 + 2r \\
&= d\big(p^{2r}\big).
\end{aligned}
$$

■

Notation For $n \geq 1$ let $g(n) = d(n^2)$. This is temporary notation for this course. Then $1 * 2^\omega = g$.

Example 3.36 $1 * g = d^2$.

Solution Since both sides are multiplicative it suffices to check equality on prime powers.

$$
\begin{aligned}
(1 * g)(p^r) &= \sum_{0 \leq b \leq r} g\big(p^b\big) = \sum_{0 \leq b \leq r} (2b+1) \\
&= 2\frac{r(r+1)}{2} + (r+1) \\
&= (r+1)^2 = d^2(p^r).
\end{aligned}
$$

■

Hence we have shown

Example 3.37

$$d^2 = 1 * 1 * 1 * 1 * \mu_2.$$

Euler's phi function

Recall the definition of Euler's phi function as

$$\phi(n) = \{1 \leq r \leq n, \, \gcd(r,n) = 1\}.$$

We 'pick out' the condition $\gcd(r,n) = 1$ using the δ function, for which $\delta(n) = 1$ if $n = 1$, zero otherwise. For then

$$\delta(\gcd(r,n)) = \begin{cases} 1 & \text{if } \gcd(r,n) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We can then use Möbius inversion, in the form $\delta(m) = \sum_{d|m} \mu(d)$ to get

Example 3.38 *Show that Euler's phi function satisfies* $\phi = \mu * j$, *i.e.*

$$\phi(n) = \sum_{d|n} \mu(d) \, \frac{n}{d}.$$

Solution

$$\begin{aligned} \phi(n) &= \sum_{\substack{r=1 \\ \gcd(r,n)=1}}^{n} 1 = \sum_{r=1}^{n} \delta\left(\gcd(r,n)\right) \qquad \text{by definition of } \delta, \\ &= \sum_{r=1}^{n} \sum_{d|\gcd(r,n)} \mu\left(d\right) \qquad \text{by Möbius inversion } \delta = 1 * \mu. \end{aligned}$$

Yet $d|\gcd(r,n)$ if, and only if, $d|r$ and $d|n$. Continuing

$$= \sum_{r=1}^{n} \sum_{\substack{d|r \\ d|n}} \mu\left(d\right) = \sum_{d|n} \mu\left(d\right) \sum_{\substack{r=1 \\ d|r}}^{n} 1.$$

on interchanging the summations. In this double summation we have that $d|n$, so $n = \ell d$ say, and we also have $d|r$, so $r = kd$ say. Thus in the inner sum we are counting the number of $k \geq 1$ for which $r \leq n$, i.e. $kd \leq \ell d$, that is, $k \leq \ell$. There are $\ell = n/d$ such values. Therefore this inner summation equals n/d and thus

$$\phi(n) = \sum_{d|n} \mu(d) \, \frac{n}{d}.$$

(Make sure you understand why this inner sum is *exactly* n/d).

■

Corollary 3.39 *1. ϕ is multiplicative.*

2.
$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

3.
$$\sum_{d|n} \phi(d) = n.$$

Proof 1. Since μ and j are multiplicative we conclude that $\phi = \mu * j$ is multiplicative.

2. Looking at ϕ on prime powers

$$\begin{aligned} \phi(p^a) &= \sum_{d|p^a} \mu(d) \frac{p^a}{d} = \sum_{0 \le k \le a} \mu\big(p^k\big)\, p^{a-k} \\ &= \sum_{0 \le k \le 1} \mu\big(p^k\big)\, p^{a-k} \quad \text{since } \mu\left(p^k\right) = 0 \text{ for all } k \ge 2, \\ &= p^a - p^{a-1}. \end{aligned}$$

This actually should have been obvious from the definition, the only natural numbers $\le p^a$, **not** coprime to p^a are the multiples of p of which there are p^{a-1} in number. So the number of natural numbers $\le p^a$, **coprime** to p^a is the difference $p^a - p^{a-1}$.

Thus, since ϕ is multiplicative,

$$\phi(n) = \prod_{p^a \| n} \phi(p^a) = \prod_{p^a \| n} \left(p^a - p^{a-1}\right) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

3. Start from $\phi = \mu * j$ and convolute both sides with 1,

$$\begin{aligned} 1 * \phi &= 1 * (\mu * j) \\ &= (1 * \mu) * j \qquad \text{since } * \text{ is associative} \\ &= \delta * j \qquad \text{by Mobius inversion, } 1 * \mu = \delta \\ &= j, \qquad \text{since } \delta \text{ is the identity under } *. \end{aligned}$$

Then, by the definition of convolution, $1 * \phi = j$ means

$$\sum_{d|n} \phi(d) = \sum_{d|n} \phi(d)\, 1\Big(\frac{n}{d}\Big) = j(n) = n.$$

■

Finally, we saw an important arithmetic function earlier in the course, namely von Mangoldt's function $\Lambda(n)$ defined to be $\log p$ when n is a power of the prime p, zero otherwise. We have not studied it here because it is not multiplicative.

The important result of Λ was

$$\sum_{d|n} \Lambda(d) = \log n, \tag{15}$$

which was introduced without motivation. But where did it come from?

If we write $\ell(n) = \log n$ we can see that the result is the convolution $\Lambda * 1 = \ell$. Then formally we can consider

$$\begin{aligned} D_{\Lambda*1}(s) &= D_\Lambda(s)\, D_1(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\zeta(s) = -\frac{\zeta'(s)}{\zeta(s)}\zeta(s) \\ &= -\zeta'(s) = \sum_{n=1}^{\infty} \frac{\log n}{n^s} \\ &= D_\ell(s)\,. \end{aligned}$$

This suggests $\Lambda * 1 = \ell$, i.e. (15). Möbius inversion applied to (15) gives $\Lambda = \mu * \ell$, i.e.

$$\Lambda(n) = \sum_{d|n} \mu(d) \log\left(\frac{n}{d}\right).$$